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Approach to coefficient Inequality for a new subclass of Starlike Function with extremals

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Abstract- The aim of the present paper is to investigate a certain subclass $S^*(A,B,p,\delta)$ of starlike function and obtain the sharp upper bound of the functional $\left|a_3-\mu a_2^2\right|$ for the analytic function $f(z)=z+a_2z^2+a_3z^3+\cdots,\qquad |z|<1$ belonging to this subclass of starlike function.

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1 INTRODUCTION

Let A denote the class of functions of the form

$$f(z)=z+a_2z^2+a_3z^3+\cdots \tag{1}$$

Which are analytic in the open unit disc $U=z:z\in C;|z|<1$ and let S denote the class of functions in A that are univalent in U .

In 1916, for the functions $f(z)\in S$, Bieber Bach [4, 5] proved the result $|a_2|\le 2$. In 1923, for the same functions, Lowner [2] proved that $|a_3|\le 3$. With these results $|a_2|\le 2$ and $|a_3|\le 3$, for the class S it was very easy to draw out the relation between a_3 and a_2^2. With the help of Lowner's method , Fekete and Szego [6] proved the following well known resul

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}3-4\mu & \text{if}\quad \mu\le 0\\ 1+\exp\left(\dfrac{-2\mu}{1-\mu}\right) & \text{if}\quad 0\le\mu\le 1\\ 4\mu-3 & \text{if}\quad \mu\ge 1\end{cases}$$

This inequality is very much helpful in determining estimates of higher coefficients for some subclasses S (See Chhichra [1], Babalola [3]).

Now we define some subclasses of S. Let

$$f(z)=z+a_2z^2+a_3z^3+\cdots\in A$$

and satisfy the condition

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>0,\quad z\in U$$

is univalent starlike function and denoted by

S^* and a subclass $S*(A,B)=\left\{f(z)\in A, \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}\right.$ where $-1\le B<A\le 1, z\in U$

It is obvious that $S^*(A,B)$ is a subclass of S^*.

We introduce a new class as

$$S^*(p)=\left\{f(z)\in A, \frac{zf'(pz)}{pf(z)} \prec \frac{1+z}{1-z}, z\in U\right\}$$

Symbol $\prec$ stands for subordination.

Analytic bounded functions: Class of analytic bounded function is of the form

$$w(z)=\sum_{n=1}^{\infty} c_n z^n, w(0)=0, |w(z)|\le 1.$$

It is known that $|c_1|\le 1, |c_2|\le 1-|c_1|^2$.

2 FEKETE-SZEGO PROBLEM

Our main result is the following

2.1 Theorem

Let the bounded function $w(z)=c_1z+c_2z^2+\cdots$ and $f(z)\in S^*(A,B,p,\delta)$, then

$$|a_2-\mu a_3|\le \begin{cases} \frac{(A-B)}{(2p-1)}\left[\frac{U+V}{2(3p^2-1)(A-B)}-\frac{(A-B)}{(2p-1)}\mu\right] \\ \qquad if\ \ \mu\le\lambda_1; \\ \frac{\delta(A-B)}{3p^2-1} \\ \qquad if\ \ \lambda_1<\mu<\lambda_2 \\ \frac{(A-B)}{(2p-1)}\left[\frac{(A-B)}{(2p-1)}\mu-\frac{U+V}{2(3p^2-1)(A-B)}\right] \\ \qquad if\ \ \mu\ge\lambda_2 \end{cases}$$

where

$$U=(A-B)(-4Bp\delta+A(3-\delta)),$$
$$V=\delta(\delta-1)(2AB+2A^2p)-(\delta(\delta-1))B^2$$

$$\lambda_1=\frac{(2p-1)}{(3p^2-1)(A-B)^2}\begin{bmatrix}\delta(B-A)(4Bp+2(2p-1))+ \\ \delta(\delta-1)((2AB+2A^2p)-B^2)+ \\ (3-\delta)A^2-2AB\end{bmatrix}$$

and

$$\lambda_2=\frac{(2p-1)}{(3p^2-1)(A-B)^2}\begin{bmatrix}\delta(A-B)(4Bp-2(2p-1))- \\ \delta(\delta-1)((2AB+2A^2p)-B^2)- \\ (3-\delta)A^2+2AB\end{bmatrix}$$

The results are sharp.

Proof: By definition of $S^*(p)$, we have

$$\frac{zf'(pz)}{pf(z)} \prec \left(\frac{1+Aw(z)}{1+Bw(z)}\right)^{\delta} \qquad \ldots(2)$$

By expending the series (2)

$$1+(2p-1)a_2z+\left((1-2p)a_2^2+(3p^2-1)a_3\right)z^2+\cdots$$
$$=1+\delta(A-B)c_1z+$$
$$\begin{pmatrix}\delta(A-B)c_2+\delta(B^2-AB)c_1^2\\ +\dfrac{\delta}{2}(\delta-1)(A-B)^2c_1^2\end{pmatrix}z^2+\cdots \qquad \ldots(3)$$

Comparing coefficients of (3)

$$a_2=\frac{\delta(A-B)c_1}{2p-1}$$ and

$$a_3=\frac{(A-B)\delta}{(3p^2-1)}c_2+\frac{\delta c_1^2}{(3p^2-1)(2p-1)}\begin{bmatrix}4Bp(B-A)\\+(\delta-1)\left(2A(B+Ap)-B^2\right)\\+(3-\delta)A^2-2AB\end{bmatrix} \qquad \ldots(4)$$

$$\left|a_3-\mu a_2^2\right|\le\frac{\delta(A-B)}{(3p^2-1)}|c_2|+\left\{\begin{matrix}\dfrac{\begin{matrix}4Bp(B-A)\\+(\delta-1)\left(2A(B+Ap)-B^2\right)\\+(3-\delta)A^2-2AB\end{matrix}}{(3p^2-1)(2p-1)}\\-\dfrac{(A-B)^2}{(2p-1)^2}\mu\end{matrix}\right\}\left|c_1^2\right|\delta^2$$

$$=\frac{\delta(A-B)}{3p^2-1}+\left\{\begin{matrix}\left|\dfrac{\begin{matrix}4Bp(B-A)\\+(\delta-1)\left(2A(B+Ap)-B^2\right)\\+(3-\delta)A^2-2AB\end{matrix}}{(3p^2-1)(2p-1)}-\dfrac{(A-B)^2}{(2p-1)^2}\mu\right|\\-\dfrac{(A-B)}{3p^2-1}\end{matrix}\right\}|c_1|^2 \qquad \ldots(5)$$

Case 1: when

$$\mu\le\frac{(2p-1)}{2(3p^2-1)(A-B)^2}\begin{bmatrix}4Bp\delta(B-A)\\+(\delta-1)\left(2A(B+Ap)-B^2\right)\\+(3-\delta)A^2-2AB\end{bmatrix}$$

Inequality (5) can be rewritten as

$$\left|a_3-\mu a_2^2\right|\le\frac{\delta(A-B)}{3p^2-1}+\left\{\begin{matrix}\left|\dfrac{\begin{matrix}4Bp(B-A)\\+(\delta-1)\left(2A(B+Ap)-B^2\right)\\+(3-\delta)A^2-2AB\end{matrix}}{(3p^2-1)(2p-1)}-\dfrac{(A-B)^2}{(2p-1)^2}\mu\right|\\-\dfrac{(A-B)}{3p^2-1}\end{matrix}\right\}|c_1|^2 \qquad \ldots(6)$$

Sub case 1(a): When

$$\mu \leq \frac{(2p-1)}{(3p^2-1)(A-B)^2}\begin{bmatrix}\delta(B-A)(4Bp+2(2p-1))+\\ \delta(\delta-1)((2AB+2A^2p)-B^2)+\\ (3-\delta)A^2-2AB\end{bmatrix},$$

Then equation (6) can be rewritten as

$$|a_3-\mu a_2^2| \leq \frac{\delta(A-B)}{(2p-1)}\left[\frac{U+V}{2(3p^2-1)(A-B)}-\frac{(A-B)}{(2p-1)}\mu\right] \quad \ldots(7)$$

s

Sub case 1(b): When

$$\frac{(2p-1)}{(3p^2-1)(A-B)^2}\begin{bmatrix}\delta(B-A)(4Bp+2(2p-1))+\\ \delta(\delta-1)((2AB+2A^2p)-B^2)+\\ (3-\delta)A^2-2AB\end{bmatrix}, < \mu < \frac{(2p-1)}{2(3p^2-1)(A-B)^2}\begin{bmatrix}4Bp\delta(B-A)\\ +(\delta-1)(2A(B+Ap)-B^2)\\ +(3-\delta)A^2-2AB\end{bmatrix}$$

then the equation (6) becomes

$$|a_3-\mu a_2^2| \leq \frac{\delta(A-B)}{3p^2-1} \quad \ldots(8)$$

Case 2: When

$$\mu \geq \frac{(2p-1)}{2(3p^2-1)(A-B)^2}\begin{bmatrix}4Bp\delta(B-A)\\ +(\delta-1)(2A(B+Ap)-B^2)\\ +(3-\delta)A^2-2AB\end{bmatrix}$$

$$|a_3-\mu a_2^2| \leq \frac{\delta(A-B)}{(2p-1)}\left[-\frac{U+V}{2(3p^2-1)(A-B)}+\frac{(A-B)}{(2p-1)}\mu\right] \quad \ldots(9)$$

Sub case 2(a): When

$$\mu \geq \frac{(2p-1)}{(3p^2-1)(A-B)^2}\begin{bmatrix}\delta(A-B)(4Bp-2(2p-1))-\\ \delta(\delta-1)((2AB+2A^2p)-B^2)-\\ (3-\delta)A^2+2AB\end{bmatrix}$$

Then the equation (9) becomes

$$|a_3-\mu a_2^2| \leq \frac{(A-B)}{(2p-1)}\left[\frac{(A-B)}{(2p-1)}\mu-\frac{U+V}{2(3p^2-1)(A-B)}\right] \quad \ldots(10)$$

Sub case 2(b): When

$$\frac{(2p-1)}{2(3p^2-1)(A-B)^2}\begin{bmatrix}4Bp\delta(B-A)\\+(\delta-1)(2A(B+Ap)-B^2)\\+(3-\delta)A^2-2AB\end{bmatrix}$$

$$<\mu<$$

$$\frac{(2p-1)}{(3p^2-1)(A-B)^2}\begin{bmatrix}\delta(A-B)(4Bp-2(2p-1))-\\\delta(\delta-1)((2AB+2A^2p)-B^2)\\-(3-\delta)A^2+2AB\end{bmatrix},$$

Then the equation (9) becomes

$$\left|a_3-\mu a_2^2\right|\le\frac{\delta(A-B)}{3p^2-1}\qquad\ldots(11)$$

Combining the equations (7), (8), (10) and (11).

We get the Fekete Szego inequality for $S^*(A,B,p,\delta)$ as

$$\left|a_2-\mu a_3\right|\le\begin{cases}\dfrac{(A-B)}{(2p-1)}\begin{bmatrix}\dfrac{U+V}{2(3p^2-1)(A-B)}\\-\dfrac{(A-B)}{(2p-1)}\mu\end{bmatrix} & if\ \ \mu\le\lambda_1;\\ \dfrac{\delta(A-B)}{3p^2-1} & if\ \ \lambda_1<\mu<\lambda_2\\ \dfrac{(A-B)}{(2p-1)}\left[\dfrac{(A-B)}{(2p-1)}\mu-\dfrac{U+V}{2(3p^2-1)(A-B)}\right] & if\ \ \mu\ge\lambda_2\end{cases}$$

The extremal function for first and third inequality is

$$f_1(z)=z\{1+az\}^n$$

Where

$$a=\frac{(A-B)^2(3p^2-1)-(U+V)(2p-1)}{(A-B)(3p^2-1)(2p-1)}$$

And extremal function for second inequality is

$$f_2(z)=z\{1+\delta(A-B)z^2\}^{\frac{1}{3p^2-1}}$$

Corollary 1: Putting $A=1,B=-1,\delta=1$ in the theorem 2.1 we get

$$\left|a_2^2-\mu a_3\right|\le\begin{cases}\dfrac{2(2p+1)}{(3p^2-1)(2p-1)}-\dfrac{4}{(2p-1)^2}\mu & if\quad \mu\le\dfrac{2p-1}{3p^2-1};\\ \dfrac{2}{3p^2-1} & if\quad \dfrac{2p-1}{3p^2-1}\le\mu\le\dfrac{2p(2p-1)}{3p^2-1}\\ \dfrac{4}{(2p-1)^2}-\dfrac{2(2p+1)}{(3p^2-1)(2p-1)} & if\quad \mu\ge\dfrac{2p(2p-1)}{3p^2-1}\end{cases}$$

Which is the result obtained by [9].

Corollary 2: Putting $p=1$ and $A=1,B=-1,\delta=1$ in the theorem 2.1 we get

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}3-4\mu & if\quad \mu\le\dfrac{1}{2};\\ 1 & if\quad \dfrac{1}{2}<\mu<1;\\ 4\mu-3 & if\quad \mu\ge1\end{cases}$$

Which is the result obtained by [8].

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